

# SPINORIAL FANO MANIFOLDS

(JOINT WORK WITH L. MANIVEL)

/C

## § CLASSIFICATION OF FANO MANIFOLDS

DEF  $X$  IS FANO IF IS A COMPACT COMPLEX MANIFOLD WITH  
 $-K_X$  AMPLE

REM BY KOHAIKI VANISHING  $b_2(X) = \rho(X) = \text{rk}(\text{Pic}(X))$   $X$  FANO  
 $\leadsto \rho(X)$  IS A DEFORMATION INVARIANT FOR FANOS

FACT [KOHAIKI - MIYAOKA - MORI '92]

IN ANY DIMENSION THERE ARE FINITELY MANY FANOS  
UP TO DEFORMATION.

UP TO DIM 3  $\rightarrow$  COMPLETE CLASSIFICATION [MUKAI - MORI]

WHAT ABOUT  $\rho(X) = 1$ ? PRIME FANOS

DIM 1  $\mathbb{P}^1 \hookrightarrow \text{SL}(2)$

DIM 2  $\mathbb{P}^2 \hookrightarrow \text{SL}(3)$

DIM 3  $\mathbb{P}^3$ ,  $\mathbb{Q}_3$ ,  $\text{Gr}(2, V_5) \cap \mathbb{P}^6 \subseteq \mathbb{P}^9$  + [14 OTHER FANOS]  
 $\uparrow \quad \quad \uparrow \quad \quad \uparrow$   
 $\text{SL}(4) \quad \text{SO}(5) \quad \text{SL}(2)$

GOAL CLASSIFICATION OF PRIME FANOS LOCALLY RIGID  
AND QUASI-HOMOGENEOUS

# § HOMOGENEOUS AND QUASI-HOMOGENEOUS VARIETIES

**DEF**  $X$  IS HOMOGENEOUS IF  $\exists G \curvearrowright X$  TRANSITIVE.  
SIMPLE COMPLEX LIE GROUP

$Y$  IS Q-HOMOGENEOUS IF  $\exists H \curvearrowright Y$  WITH AN ORBIT (DEGREE) ORBIT.

## EXAMPLES

1)  $\mathbb{P}^n \hookrightarrow SL(n+1)$  TRANSITIVE

2)  $Gr(k, V_n) \hookrightarrow SL(n)$  TRANSITIVE

3)  $OGr(k, V_m) \hookrightarrow SO(m) \xleftarrow{1:2} SPIN(m)$  TRANSITIVE

$$\left\{ U_k \in (V_m, q) \mid q|_{U_k} = 0 \right\}$$

ii  
POS-DEFINITE  
QUADRATIC FORM

LEM ON  $OGr(k, V_m)$  THERE EXISTS A (HOMOGENEOUS) VECTOR BUNDLE

$S$ , SPINOR BUNDLE S.t.

$$M = 2n+1$$

• (g.g.)  $H^0(OGr, S) \simeq \Delta$   
SPINOR REPRESENTATION

$$\Delta \simeq \wedge^n E_n \quad \dim \Delta = 2^n$$

$$V_m = (E_n \oplus F_n)^\perp \oplus \langle V_m \rangle$$

$$q = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

• THE FIBER OF  $S$  OVER  $U_k$  IS

$$\text{THE SPINOR REPRESENTATION OF } \left( \frac{U_k^\perp}{U_k}, \tilde{q} \right)$$

$$2n+1-k-k = 2(n-k)+1$$

$$\Rightarrow \text{rk}(S) = 2^{n-k}$$

• THE FIRST CHERN CLASS OF  $S$  IS  $C_1(S) = 2^{n-k-1} L$   
( $k < n$ )

$$\text{WHERE } \text{PIC}(OGr(k, V_m)) = \mathbb{Z}L$$

## PROCEDURE TO CONSTRUCT SPINORAL FANOS

$$\text{LET } [S] \in \mathbb{P}(\Delta) \simeq \mathbb{P}(H^0(OGr, S)) \ni [\Delta_S]$$

$$Y := Z(\Delta_S) \subseteq OGr(k, V_m)$$

[SATO-KIMURA '77] CLASSIFICATION OF ORBITS OF  $SPIN(m) \curvearrowright \mathbb{P}(\Delta)$

FOR  $m \leq 14$ ,  $m \neq 13$  THERE EXISTS AN ORBIT ORBIT  $\emptyset$  ✓

THEY COMPUTE THE STABILIZER OF  $[S] \in \emptyset$

$$\text{LEM } S_1, S_2 \in \emptyset \Rightarrow Z(\Delta_{S_1}) = Z(\Delta_{S_2})$$

$$\text{NOTATION } Y = Z(\Delta_S) \quad Y = (OGr(k, V_m), S)$$

$$\text{LEM } g \in H = \text{stab} \left( \begin{smallmatrix} S \\ \uparrow \\ \emptyset \end{smallmatrix} \right) \subseteq SPIN(m)$$

$$x \in Z(\Delta_S)$$

$$\Delta_S(g \cdot x) = g^{-1} \Delta_S(x) = 0$$

$$\text{THM (F-N)} \text{ FOR } m \leq 14 \quad m \neq 13 \quad Y = (OGr(k, V_m), S)$$

IS PRIME FAN  $\emptyset$  AND QUASI-HOMOGENEOUS  
(WITH FEW EXCEPTIONS)

THM (F-N) ALL OF THEM ARE LOCALLY REG.  
( $H^1(Y, T_Y) = 0$ )

# § CLASSIFICATION TABLE

|        |       |                                                                                                                                       |                            |                        |                                                                                                                                                          |
|--------|-------|---------------------------------------------------------------------------------------------------------------------------------------|----------------------------|------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------|
| $M=9$  | $B_4$ | $(OG(2,9), S) \simeq OG(2,7)$<br>$(OG(3,9), S)$                                                                                       | 7<br>10                    | 4<br>4                 | "Aut"<br>SPIN(7)<br>SPIN(7)                                                                                                                              |
| $M=10$ | $D_5$ | $(OG(2,10), S_+)$<br>$(OG(3,10), S_+)$<br>$(OG(5,10)_+, S_+)$                                                                         | 9<br>13<br>9               | 5<br>5<br>7            | $(\mathbb{F}_m \times SPIN(7)) \rtimes \Delta_8$<br>$(\mathbb{F}_m \times SPIN(7)) \rtimes \Delta_8$<br>$(\mathbb{F}_m \times SPIN(7)) \rtimes \Delta_8$ |
| $M=11$ | $B_5$ | $(OG(2,11), S) \simeq FL(1,4,5)$<br>$(OG(3,11), S)$                                                                                   | 7<br>14                    | 4<br>5                 | $SL(5)$<br>$SL(5)$                                                                                                                                       |
| $M=12$ | $D_6$ | $(OG(2,12), S_+) \simeq FL(1,5,6)$<br>$(OG(3,12), S_+)$<br>$(OG(4,12), S_+)$<br>$(OG(6,12)_+, S_+)$                                   | 9<br>17<br>20<br>14        | 5<br>6<br>6<br>9       | $SL(6)$<br>$SL(6)$<br>$SL(6)$<br>$SL(6)$                                                                                                                 |
| $M=14$ | $D_7$ | $(OG(2,14), S_+) \simeq G_2/P_2 \times G_2/P_2$<br>$(OG(3,14), S_+)$<br>$(OG(4,14), S_+)$<br>$(OG(5,14), S_+)$<br>$(OG(7,14)_+, S_+)$ | 10<br>19<br>26<br>28<br>20 | 3<br>6<br>7<br>7<br>11 | $G_2 \times G_2$<br>$G_2 \times G_2$<br>$G_2 \times G_2^*$<br>$G_2 \times G_2^*$<br>$G_2 \times G_2$                                                     |

# § WORKED EXAMPLES

- $(\mathcal{O}_{Gr(2,9)}, S) \simeq \mathcal{O}_{Gr(2,7)}$

$$\underline{SK} \quad \underset{SPIN(9)}{stab(\delta)} \simeq \underset{SPIN(7)}{stab(\delta)} \quad (\text{EXPLICIT})$$

$$\left( \begin{array}{cc} \text{PASCAL - FERRIS} & '10 \\ \text{KWANG - LI} & '22 \end{array} \right) \quad \mathcal{O}_{Gr(2,7)} \quad \text{ADmits a UNIQUE} \quad \text{SMOOTH DEGENERATION} \quad \text{TO THE } G_2\text{-HOMOGENEOUS VARIETY}$$

PROP (F.M) THIS DEGENERATION IS EXPLICIT IN OUR MODEL OF  $\mathcal{O}_{Gr(2,7)}$

$$\begin{array}{c} IP(\Delta_3) = O_{15} \cup V(\tilde{Q}) \quad \text{QUADRIC (IGUSA)} \\ \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\ \quad \quad \delta_1 \quad O_{14} \cup \mathcal{O}_{Gr(4,9)} \\ \quad \quad \quad \downarrow \\ \quad \quad \delta_2 \end{array}$$

$\leadsto Z(\cap \delta_2)$  IS SMOOTH WITH  $\text{Aut} \neq \text{Spin}(7)$

$\leadsto Z(\cap \delta_2)$  IS  $G_2$ -HOMO

- $(\mathcal{O}_{Gr(3,9)}, S)$  IS  $\alpha$ -HOMOGENEOUS UNDER  $\underset{21}{\text{Spin}(7)}$

$$\dim Y = 3 \cdot 6 - 4 - 4 = \underline{10}$$

FIND  $P_3 \in V_g$  ISOTROPIC,  $\Lambda_\delta(P_3) = 0$

$$\text{stab}(P_3) \subseteq \text{Spin}(7) \quad \dim \text{stab}(P_3) = 11$$

$$\Rightarrow \dim \mathcal{O}_{P_3} = 10$$